PHYSICAL REVIEW E **69**, 026201 (2004)

Time-varying synchronization of chaotic systems in the presence of system mismatch

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(Received 21 July 2003; published 17 February 2004)

The problem of synchronization of two identical chaotic systems in the presence of system mismatch is investigated in this article. The instantaneous mean square error ($\mathcal{E}$) of the unidirectionally coupled synchronization scheme is analyzed based on the Jacobian equation of the response system. It is shown that synchronization based on a constant coupling parameter does not produce satisfactory performance. A synchronization scheme is proposed here, and the time-varying coupling parameter sequence used in this new scheme is obtained by minimizing the instantaneous $\mathcal{E}$. Numerical simulations show that the proposed time-varying synchronization method has smaller mean square synchronization error than the conventional approach based on using a constant coupling parameter.

DOI: 10.1103/PhysRevE.69.026201 PACS number(s): 05.45.Xt

I. INTRODUCTION

Since the first observation of two systems to exhibit unpredictable behavior yet evolve in perfect synchrony [1–5], this behavior has been the subject of a substantial number of investigations because of the intrinsic interest in the idea of synchronization between chaotic motions, and its potential application. Most of the research in the field of chaos synchronization assumes that the drive system is connected to the response by an ideal channel. However, there are always some mismatches between the drive and response systems in practical applications. The system mismatch, including component mismatch and channel noise, has been shown to be able to induce momentary large bursts away from synchronization for some systems, and seriously limits the application of chaos synchronization [6–8]. The robustness and stability of synchronization in the presence of these mismatches are discussed in [9–11]. It is shown that synchronization stability is related to the conditional Lyapunov exponents and transversal Lyapunov exponents of the chaotic system [12,13].

However, recent research shows that stability is inadequate to guarantee a high-quality synchronization performance. A quantitative measure is needed to characterize the performance of a synchronization scheme. The mean square error ($\mathcal{E}$) between the states of the drive and response systems is introduced in [14] to investigate the performance of a unidirectionally coupled synchronization system in the presence of channel noise. An optimal constant coupling parameter is then obtained by minimizing the mean square synchronization error. It is shown that the optimal coupling parameter does not only depend on the global Lyapunov exponents, but also depends on the local Lyapunov exponents.

The unidirectionally coupled synchronization scheme is considered in this study. Not only is it easy to implement and does not require a numerical procedure to determine the system behavior in practical applications, but it is also shown to be a generalization of the Pecora and Carroll synchronization method [15]. While the conventional coupled synchronization method always uses a constant coupling parameter, it is shown here that a time-varying coupled parameter should be used to achieve a good synchronization performance in the presence of system mismatches. In particular, we propose using the instantaneous $\mathcal{E}$ to measure the $\mathcal{E}$ performance of the Jacobian of the response system. It is found that the instantaneous $\mathcal{E}$ depends on the eigenvalues of the Jacobian matrix, which is usually time varying for a chaotic system. The time-varying nature of these eigenvalues indicates that instantaneous $\mathcal{E}$ varies with time and hence using a constant coupling parameter is insufficient to minimize the $\mathcal{E}$ in synchronization. In other words, the performance of the conventional synchronization method can be improved by using a time-varying coupling parameter sequence. Based on this observation, a new design approach for a time-varying synchronization is proposed here. The coupling parameter at a certain time is obtained by minimizing the instantaneous $\mathcal{E}$ at that time instant. When there is no system mismatch, the conventional synchronization method is found to produce the same performance as the proposed time-varying approach. But when system mismatch exists, the time-varying synchronization is superior to the optimal constant coupling parameter approach.

The remainder of this article is organized as follows. In Sec. II, we introduce the problem of chaos synchronization with system mismatches. We then analyze the instantaneous $\mathcal{E}$ of unidirectionally coupled synchronization, and propose the time-varying coupling synchronization scheme. Section III gives the computer simulation to show the synchronization performance of the proposed approach. Concluding remarks are given in Sec. IV.

II. DESIGN METHOD FOR CHAOS SYNCHRONIZATION IN NOISE

Suppose that the dynamic of the drive system is given by

$$\mathbf{x}_n = \mathbf{f}(\mathbf{x}_{n-1}), \tag{1}$$

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where $\mathbf{x}_n=[x_1(n),x_2(n),\ldots,x_d(n)]^{\mathrm{T}}$ is the d-dimensional state vector at time n, $\mathbf{f}(\mathbf{x}_n)=[f_1(\mathbf{x}_n),f_2(\mathbf{x}_n),\ldots,f_d(\mathbf{x}_n)]^{\mathrm{T}}$ are continuous nonlinear functions, and "T" denotes the transpose of a vector or matrix.

To utilize a channel efficiently, a scalar driving signal is usually transmitted for synchronization. That is,

$$y_n=\mathbf{h}^{\mathrm{T}}\mathbf{x}_n+v_n, \tag{2}$$

where $\mathbf{h}=[h_1,h_2,\ldots,h_d]^{\mathrm{T}}$, v_n is a Gaussian channel noise process with zero mean and $\mathrm{E}(v_i v_j)=R\delta_{ij}\geqslant 0$, $\mathrm{E}(\cdot)$ is the mathematical expectation operator, and δ_{ij} is the Kronecker delta function.

Chaos synchronization is to build a response system such that it will follow the states of the drive system in (1) based on the driving signal y_n. In a unidirectionally coupled synchronization scheme, the dynamic of the response system can be described by

$$\hat{\mathbf{x}}_n=\mathbf{f}(\hat{\mathbf{x}}_{n-1})+\mathbf{k}(y_n-\mathbf{h}^{\mathrm{T}}\mathbf{f}(\hat{\mathbf{x}}_{n-1})), \tag{3}$$

where $\hat{\mathbf{x}}_n=(\hat{x}_1(n),\hat{x}_2(n),\ldots,\hat{x}_d(n))^{\mathrm{T}}$ is the d-dimensional state vector of the response system, $\mathbf{k}=(k_1,k_2,\ldots,k_d)^{\mathrm{T}}$ is a d-dimensional vector which represents the coupling parameter.

In the presence of channel noise, the ideal synchronization, that is, the states of response system equal to those of drive system, cannot be fulfilled. There is always some synchronization error $\mathbf{e}_n=\mathbf{x}_n-\hat{\mathbf{x}}_n\neq\mathbf{0}$ as $n\to\infty$ in the response system. An approximated synchronization is used to describe this kind of synchronization behavior. Robustness and stability of synchronization can be reached by using a suitable coupling parameter. The following mean square error ($\mathcal{E}$) between the drive and response systems is used here to quantitatively describe the synchronization performance

$$\mathcal{E}=\lim_{n\to\infty}\frac{1}{n-n_0+1}\sum_{i=n_0}^{n}\mathbf{e}_i^{\mathrm{T}}\mathbf{e}_i, \tag{4}$$

where n_0 is the length of the transient state.

Subtracting (3) by (1), the synchronization error $\mathbf{e}_n$ can be expressed as

$$\mathbf{e}_n=\mathbf{f}(\hat{\mathbf{x}}_{n-1})-\mathbf{f}(\mathbf{x}_{n-1})+\mathbf{k}(\mathbf{h}^{\mathrm{T}}\mathbf{f}(\mathbf{x}_{n-1})-\mathbf{h}^{\mathrm{T}}\mathbf{f}(\hat{\mathbf{x}}_{n-1}))+\mathbf{k}v_n. \tag{5}$$

Suppose that when the channel noise is small, the synchronization error is small over the steady-state synchronization period. Linearizing (5) at the synchronized state $\hat{\mathbf{x}}_n$ gives

$$\mathbf{e}_n=\mathbf{F}_{n-1}\mathbf{e}_{n-1}-\mathbf{k}\mathbf{h}^{\mathrm{T}}\mathbf{e}_{n-1}+\mathbf{k}v_n=(\mathbf{I_d}-\mathbf{k}\mathbf{h}^{\mathrm{T}})\mathbf{F}_{n-1}\mathbf{e}_{n-1}+\mathbf{k}v_n, \tag{6}$$

where $\mathbf{F}_n=(\partial\mathbf{f}/\partial\mathbf{x})|_{\mathbf{x}=\hat{\mathbf{x}}_n}$ is the Jacobian of $\mathbf{f}(\mathbf{x})$ evaluated at $\hat{\mathbf{x}}_n$.

In the absence of channel noise, the noise term v_n vanishes and the synchronization of the two chaotic systems is realized once the norm $\|\mathbf{e}_n\|$ approaches 0 as $n\to\infty$ provided that an appropriate coupling parameter $\mathbf{k}$ is chosen. However, when there exists some channel noise, synchronization error is unavoidable.[16]

To describe the effect of coupling parameter $\mathbf{k}$ on the synchronization performance at time m, the instantaneous $\mathcal{E}$ of the response system is introduced. The instantaneous $\mathcal{E}$ depicts the performance of the response system at time n by analyzing another linear system whose characteristics are the same as those of the Jacobian equation at that time. That is,

$$\mathbf{e}_m=(\mathbf{I_d}-\mathbf{k}\mathbf{h}^{\mathrm{T}})\mathbf{F}_{m-1}^{n}\mathbf{e}_{m-1}+\mathbf{k}v_m, \tag{7}$$

where $\mathbf{F}_{m-1}^{n}$ has the same eigenvalues as F_{n-1}. The instantaneous $\mathcal{E}$ at time n, $\mathcal{E}_n$, is the mean square average of the steady state in (7),

$$\mathcal{E}_n=\lim_{M\to\infty}\sum_{m=1}^{M}\mathbf{e}_m^{\mathrm{T}}\mathbf{e}_m. \tag{8}$$

Expressing the instantaneous error $\mathbf{e}_m$ in (7) in terms of its past histories from $m-1$ to m_0, we have

$$\mathbf{e}_m=\mathbf{T}(m,m_0)\mathbf{e}_{m_0}+\sum_{j=m_0+1}^{m}\mathbf{T}^{n}(m,j)\mathbf{k}v_j, \tag{9}$$

where $\mathbf{T}^n(m,j)=\Pi_{i=j+1}^{m}(\mathbf{I}-\mathbf{k}\mathbf{h}^{\mathrm{T}})\mathbf{F}_{i-1}^{n}$ is the evolution operator, which satisfies the following conditions: $\mathbf{T}^n(m,m)=\mathbf{I}$ and $\mathbf{T}^n(m,j)=\mathbf{0}$ for $j>m$.

After some algebraic manipulations,[1] $\mathcal{E}_n$ in (8) can be expressed as

$$\mathcal{E}_n=\lim_{m\to\infty}R\sum_{p=1}^{d}\sum_{q=1}^{d}\sum_{r=0}^{m}\delta_p\delta_{p,q}e^{2r(\Lambda_{B,p}+\Lambda_{C,q}^{n})}, \tag{10}$$

where $\delta_p=\mathbf{k}^{\mathrm{T}}\mathbf{v}_{B,p}$, $\delta_{p,q}=\mathrm{tr}\{\mathbf{k}\mathbf{v}_{B,p}^{\mathrm{T}}\mathbf{I}_d^q\}$, $\Lambda_{B,p}=\ln(\lambda_{B,p})/2$, $\mathbf{v}_{B,p}$, and $\lambda_{B,p}$, $p=1,2,\ldots,d$, are the eigenvectors and eigenvalues of $(\mathbf{I}_d-\mathbf{k}\mathbf{h}^{\mathrm{T}})(\mathbf{I}_d-\mathbf{k}\mathbf{h}^{\mathrm{T}})^{\mathrm{T}}$, respectively, $\Lambda_{C,q}^{n}$, $q=1,2,\ldots,d$, is the eigenvalues of $\mathbf{F}_{n-1}\mathbf{F}_{n-1}^{\mathrm{T}}$, $\mathrm{tr}(\cdot)$ denotes trace of a matrix, $\mathbf{I}_d$ denotes $d\times d$ identity matrix, and $\mathbf{I}_d^q$ denotes $d\times d$ matrix whose elements are all zero except a one in the qth position of the diagonal.

If $\Lambda_{B,p}+\Lambda_{C,q}^{n}\geqslant 0$, $\mathcal{E}_n$ will tend to infinity. An approximated synchronization cannot be obtained. Thus it is necessary to make sure that $\Lambda_{B,p}+\Lambda_{C,q}^{n}$ is negative, and hence $\mathcal{E}_n$ can be simplified as

$$\mathcal{E}_n=\sum_{p=1}^{d}\sum_{q=1}^{d}\frac{R\delta_p\delta_{p,q}}{1-e^{2(\Lambda_{B,p}+\Lambda_{C,q}^{n})}}. \tag{11}$$

Equation (11) indicates that $\mathcal{E}_n$ depends on $\Lambda_{C,q}^{n}$, which is related to the eigenvalues of the Jacobian matrix of the re-

[1]The derivation is the same as that in [14] except that the $\mathbf{F}_{i-1}$ is replaced by $\mathbf{F}_{i-1}^{n}$. The derivation is independent of the type of dynamical systems. For type I systems [15], $\Lambda_{C,q}^{n}$ is a constant with different n, while type II systems have time-varying $\Lambda_{C,q}^{n}$.

sponse system at time n, $\mathbf{F}_{n-1}$. As $\mathbf{F}_{n-1}=(\partial \mathbf{f}/\partial \mathbf{x})|_{\mathbf{x}=\hat{\mathbf{x}}_{n-1}}$, that is, $\mathbf{F}_{n-1}$ is determined by the system equation and state $\hat{\mathbf{x}}_{n-1}$. As the system state of a chaotic system is an aperiodic process, the Jacobian matrix and $\mathcal{E}_n$ usually vary with time.

The $\mathcal{E}$ in (5) is basically the average of the instantaneous $\mathcal{E}$ over time. When time varying parameters that minimize $\mathcal{E}_n$ at time n are used, we should have an improved synchronization performance if the eigenvalues of the Jacobian matrix of the system are indeed time varying. From (11), the optimal coupling parameter $\mathbf{k}$ at time n is related to the eigenvalues of $\mathbf{F}_{n-1}$. Since $\mathbf{F}_{n-1}$ depends on time in general, using a single constant coupling parameter is therefore insufficient to minimize the $\mathcal{E}$ at all time.

Based on the above analysis, a synchronization system with time-varying coupling parameter is proposed. The time-varying coupling parameter is chosen to minimize the $\mathcal{E}_n$ in (11). For the special case, that $\Lambda^n_{C,q}$ in (11) are independent of $\mathbf{F}_{n-1}$, the $\mathcal{E}_n$ is equivalent to MSE, and hence using a single coupling parameter is sufficient for an optimal synchronization.

The proposed time-varying synchronization scheme is summarized as follows.

Initial conditions,

$$\hat{\mathbf{x}}_0=\mathrm{E}[\hat{\mathbf{x}}_0]. \tag{12}$$

For $n=1$ to N, we calculate the

(1) Jacobian matrix, $\mathbf{F}_{n-1}$,
(2) eigenvectors and eigenvalues of $\mathbf{F}_{n-1}\mathbf{F}^{\mathrm{T}}_{n-1}$,
(3) minimization of the $\mathcal{E}_n$ with respect to the $\mathbf{k}$ to obtain an optimal coupling parameter, $\mathbf{k}_n$, and
(4) state vector estimation according to $\hat{\mathbf{x}}_n=\mathbf{f}(\hat{\mathbf{x}}_{n-1})+\mathbf{k}_n(y_n-\mathbf{h}^{\mathrm{T}}\mathbf{f}(\hat{\mathbf{x}}_{n-1}))$.

It should be noted that $\mathcal{E}_n$ has a linear relationship with the noise variance R in (11). If there is no channel noise, i.e., $R=0$, then $\mathcal{E}_n$ is equal to zero if the coupling parameter satisfies the condition: $\Lambda_{B,p}+\Lambda^n_{C,q}<0$. No matter what kind of parameter, constant or time varying, is used, $\mathcal{E}_n$ and $\mathcal{E}$ will be the same and equal to zero. It is why using a constant coupling parameter is sufficient for the noise-free channel in conventional synchronization.

The method we proposed here can also be applied to continuous-time dynamical system. Using the similar derivation, a linear differential equation about the synchronization error corresponding to Eq. (6) can be obtained for a continuous-time dynamical system. A state transition matrix is used to express its solution, which can be transformed as a discrete-time map. The time-varying coupling parameter can then be obtained.

III. NUMERICAL SIMULATIONS

In this section, the proposed design method is applied to three chaos synchronization systems: the tent map, logistic map, and henon map. In the following simulations, the $\mathcal{E}$ in (4) is estimated by averaging the mean square synchronization error over 20 Monte Carlo trials with different initial conditions. The synchronization error for each trial is obtained using 10^5 time points after a transient state period of $N_0=10^3$.

A. The tent map

The tent map is given by $x_n=f_t(x_{n-1},a)=a(1-|2x_{n-1}-1|)$, where $a\in[0,1]$ and $x_n\in[0,1]$. The response system of the unidirectionally coupling synchronization scheme for the tent map is then given by

$$\hat{x}_n=f_t(\hat{x}_{n-1},a)+k_n(y_n-f_t(\hat{x}_{n-1},a)), \tag{13}$$

where k_n is the coupling parameter sequence obtained by minimizing the $\mathcal{E}_n$ in (11). Without loss of generality, we use $\mathbf{h}=1$ in (2) and $y_n=x_n+v_n$.

The $\mathcal{E}_n$ for the tent map can be expressed as

$$\mathcal{E}_n=\frac{Rk_n^2}{1-F^2_{t,n-1}(1-k_n)^2}, \tag{14}$$

where $F_{t,n-1}=2a\,\mathrm{sgn}(1/2-\hat{x}_{n-1})$.

Minimizing the $\mathcal{E}_n$ in (14) with respect to k_n, the optimal coupling parameter sequence, k_n, can be obtained

$$k_n=\frac{4a^2-1}{4a^2}. \tag{15}$$

For the tent map, the $\mathcal{E}_n$ is independent of the $\hat{x}_{n-1}$, the coupling parameter designed by using the proposed method is therefore a constant, which is equal to the optimal coupling parameter [14]. Thus, the proposed method is equivalent to the optimal constant coupling parameter method for the tent map.

B. The logistic map

The logistic map is defined as $x_n=f_\ell(x_{n-1})=4x_{n-1}(1-x_{n-1})$, where $x_n\in[0,1]$. The response system in the unidirectionally coupling synchronization scheme for the logistic map is then given by

$$\hat{x}_n=f_\ell(\hat{x}_{n-1})+k_n(y_n-f_\ell(\hat{x}_{n-1})), \tag{16}$$

where k_n is the coupling parameter sequence which is obtained by minimizing the $\mathcal{E}_n$ in (11). Without loss of generality, we set $\mathbf{h}=1$ in (2) and hence $y_n=x_n+v_n$.

The instantaneous $\mathcal{E}$ for the logistic map can be expressed as

$$\mathcal{E}_n=\frac{Rk_n^2}{1-F^2_{\ell,n-1}(1-k_n)^2}, \tag{17}$$

where $F_{\ell,n-1}=4(1-2\hat{x}_{n-1})$.

Minimizing the $\mathcal{E}_n$ in (17) with respect to k_n, the optimal coupling parameter sequence, k_n, for the logistic map is given by

$$k_n=\frac{F^2_{\ell,n-1}-1}{F^2_{\ell,n-1}}. \tag{18}$$

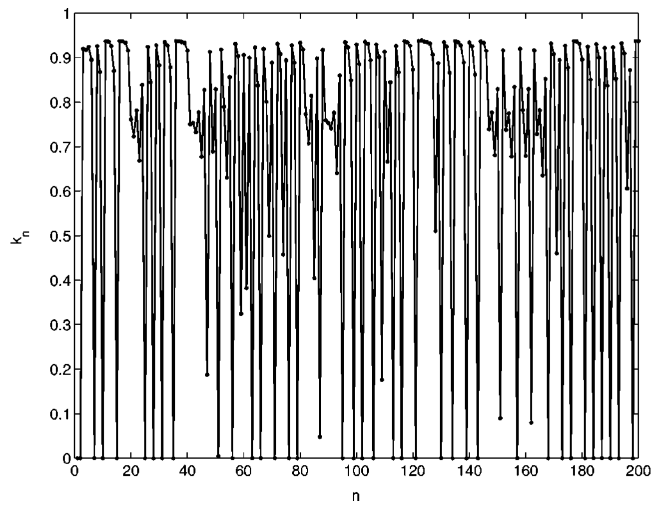

FIG. 1. An example of the time-varying coupling parameter sequence for the logistic map.

When $F^2_{\ell,n-1}$ is less than 1, the coupling parameter at time n is negative. It should be noted that it is necessary to have a non-negative coupling parameter to eliminate the effect of the initial condition error between the response and drive systems [14]. k_n is therefore set to zero when $F^2_{\ell,n-1}<1$.

Figure 1 gives an example of the time-varying coupling parameter sequence designed by using the proposed synchronization method for the logistic map. The synchronization performance versus channel noise variance is shown in Fig. 2. For comparison, the synchronization performance of the response system with the optimal constant coupling parameter is also given. The ratio of the output noise variance to the input noise variance is used in the figure to show the improvement. For the constant coupling parameter system, the parameter value of 0.88 is shown to be optimal [14], and is used in our simulations. Figure 2 shows that the synchronization system with the proposed design method always has

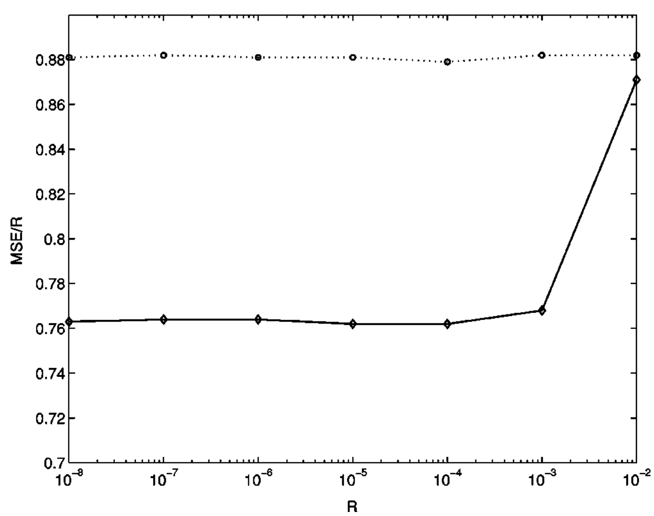

FIG. 2. Performance comparison of the unidirectionally coupling synchronization systems based on the logistic map using the optimal constant parameter method (dotted line with ○) and the proposed time-varying synchronization method (solid line with ◇).

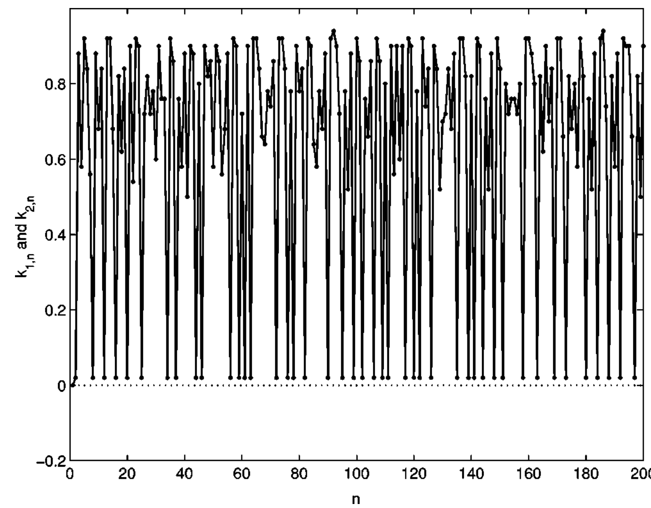

FIG. 3. An example of the time-varying coupling parameter sequence for the henon map (solid line with ◇ for k_1; dotted line for k_2).

a smaller $\mathcal{E}$ than the optimal constant coupling parameter system for all levels of channel noise. While the $\mathcal{E}/R$ of the optimal coupling parameter is 0.88, the $\mathcal{E}/R$ of the proposed method is about 0.76 as the variance of channel is smaller than 10^{-3}.

C. The henon map

The henon map is given by

$$x_{1,n}=1-1.4x^2_{1,n-1}+x_{2,n-1}, \tag{19}$$

$$x_{2,n}=0.3x_{1,n-1}. \tag{20}$$

In this case, $\mathbf{x}_n=[x_{1,n},x_{2,n}]^{\mathrm{T}}$, $\hat{\mathbf{x}}_n=[\hat{x}_{1,n},\hat{x}_{2,n}]^{\mathrm{T}}$, and $\mathbf{k}_n=[k_{1,n},k_{2,n}]$.

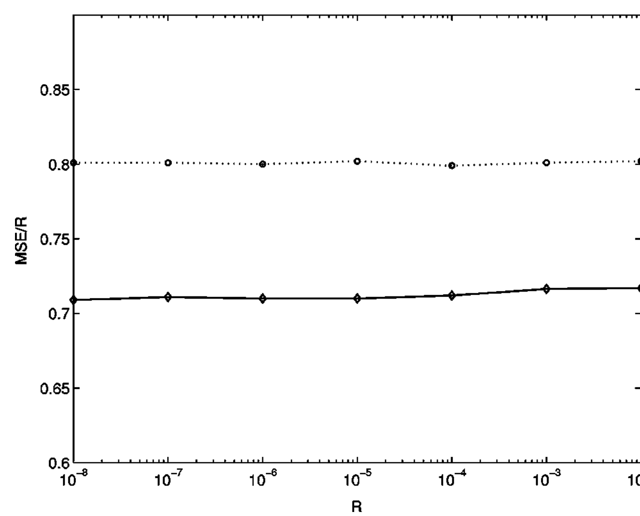

FIG. 4. Performance comparison of the unidirectionally coupling synchronization systems based on the henon map using the optimal constant parameter method (dotted line with ○) and the proposed time-varying synchronization method (solid line with ◇).

Let $\mathbf{f}_h$ denotes the henon map, i.e., $\mathbf{x}_n = \mathbf{f}_h(\mathbf{x}_{n-1})$. The response system for the henon map can be expressed as

$$\hat{\mathbf{x}}_n = \mathbf{f}_h(\hat{\mathbf{x}}_{n-1}) + \mathbf{k}_n(y_n - \mathbf{h}^{\mathrm{T}}\mathbf{f}_h(\hat{\mathbf{x}}_{n-1})), \tag{21}$$

where $\hat{\mathbf{x}}_n = [\hat{x}_{1,n}, \hat{x}_{2,n}]^{\mathrm{T}}$, $\mathbf{k}_n = [k_{1,n}, k_{2,n}]^{\mathrm{T}}$ is the coupling parameter sequence, and $\mathbf{h} = [h_1, h_2]^{\mathrm{T}}$. Here, $\mathbf{h} = [1,0]^{\mathrm{T}}$ is used, i.e., $y_n = x_{1,n} + v_n$.

Figure 3 gives an example of the time-varying coupling parameter sequence for the henon map. Both parameter sequences are obtained by minimizing the $\mathcal{E}_n$ in (11). The synchronization performance versus the noise variance is shown in Fig. 4, and the performance of the optimal coupling parameter synchronization is also plotted for comparison. For the constant coupling parameter system, the parameters $k_{1,n} = 0.88$ and $k_{2,n} = 0$ are used here which have been shown to be optimal [14]. Figure 4 shows that the synchronization system with the coupling parameter using the proposed time-varying design method has smaller $\mathcal{E}$ than the optimal coupling parameter system for all noise variance levels. The $\mathcal{E}/R$ of the optimal coupling parameter is about 0.8, while the $\mathcal{E}/R$ of the proposed method is about 0.71.

IV. CONCLUSION

The problem of synchronization of two identical chaotic systems in the presence of system mismatches is investigated here. Based on the analysis of the instantaneous $\mathcal{E}$ of the Jacobian, a novel synchronization approach with time-varying coupling parameter is developed here. Compared to the constant parameter synchronization system, the $\mathcal{E}$ between the response and drive systems is shown to be reduced by using a time-varying coupling parameter sequence that minimizes the instantaneous $\mathcal{E}$.

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